The Generalized Springer Correspondence

Amber Russell
Butler University

Basics of Representation Theory

- A group representation is a group action on a vector space, so that each group element defines a linear transformation.
- A representation is irreducible if there are no subspaces invariant under all of the linear transformations.

Example: Consider the symmetric group S_3. We can define a representation on $\mathbb{C}^3$ by permuting the standard basis vectors. However, this representation is not irreducible. The subspace given by $Span\{(1,1,1)\}$ is invariant under each of these permutation transformations.

The irreducible representations of S_n are parametrized by partitions of n.

Lie Theory Background

Notation and Examples

G - a complex algebraic group
Ex: $SL_n(\mathbb{C}) := \{n \times n$ complex matrices with det=1$\}$
$\mathfrak{g}$ - the Lie algebra associated to G
Ex: $\mathfrak{sl}_n(\mathbb{C}) := \{n \times n$ complex matrices with trace 0$\}$
$\mathcal{W}_G$ - the Weyl group of G
Ex: $\mathcal{W}_{SL_n(\mathbb{C})} :=$ the symmetric group S_n
$\mathcal{N}_{\mathfrak{g}}$ - the nilpotent cone, a variety formed from the nilpotent elements in $\mathfrak{g}$
Ex: $\mathcal{N}_{\mathfrak{sl}_n} := \{$nilpotent matrices with trace 0$\}$

Action of G

- The algebraic group G acts on its Lie algebra $\mathfrak{g}$ by the "adjoint action."
- When G and $\mathfrak{g}$ are sets of matrices, the action is by conjugation of matrices.
- The group G also acts on $\mathcal{N}_{\mathfrak{g}}$, and this action has finitely many orbits. We'll denote these orbits by $\mathcal{O}_x$ where $x \in \mathcal{N}_{\mathfrak{g}}$ is a particular nilpotent element in the orbit.

They also correspond to partitions of n.

The Springer Correspondence

Irreducible Representations of $\mathcal{W}_G$ $\longrightarrow$ Pairs $(\mathcal{O}_x, \varphi)$ where φ is an irreducible representation of $\pi_1(\mathcal{O}_x)$

For $SL_n(\mathbb{C})$, the φ are all trivial, and the correspondence is just between nilpotent orbits and representations of S_n.

The Springer Resolution and the Springer Fibers

The nilpotent cone is not a smooth variety, and a resolution of its singularities is given by

$$\mu : \widetilde{\mathcal{N}} \to \mathcal{N}$$

where $\widetilde{\mathcal{N}} = \{(x, B)|$ where B is a Borel subgroup of G and $x \in \mathcal{N} \cap \mathrm{Lie}B\}$. This $\widetilde{\mathcal{N}}$ can also be seen as the cotangent bundle of the flag variety G/B.

The fibers $\mu^{-1}(x)$ of this map are called Springer Fibers. While the Weyl group $\mathcal{W}_G$ does not act on $\mu^{-1}(x)$, it does act on the cohomology space $H^{top}(\mu^{-1}(x))$, as does $\pi_1(\mathcal{O}_x)$. Understanding these actions is what led to Springer's proof of his correspondence. Combining this with the discovery of perverse sheaves led to the alternate proof by Borho and MacPherson.

Borho–MacPherson Proof of the Springer Correspondence

Let $\underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}$ denote the constant sheaf on $\widetilde{\mathcal{N}}$. Then the Decomposition Theorem for Perverse Sheaves says

$$R\mu_* \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}} = \bigoplus_{x,\varphi} \mathrm{IC}(\mathcal{O}_x, L_\varphi) \otimes V_{x,\varphi}$$

where $\mathrm{IC}(\mathcal{O}_x, L_\varphi)$ is a simple perverse sheaf, $V_{x,\varphi}$ is a representation of $\mathcal{W}_G$, and the (x, φ) are the pairs appearing in the Springer Correspondence.

Something's Missing...

While all irreducible representation of $\mathcal{W}_G$ are accounted for in the Springer Correspondence, not all possible pairs $(\mathcal{O}_x, \varphi)$ appear. In $SL_n(\mathbb{C})$, for example, $\pi_1(\mathcal{O}_x) = \mathbb{Z}_d$ where d is the greatest common divisor of the parts in the partition corresponding to $\mathcal{O}_x$, but only the trivial representations of $\pi_1(\mathcal{O}_x)$ appear in the Springer Correspondence.

A New Approach

While Lusztig's proof relies on the theory of perverse sheaves, it does not use the Decomposition Theorem in the same way as the Borho–MacPherson result. A new approach that does resemble the Borho–MacPherson result, but only holds when G is $SL_n(\mathbb{C})$ or something similar, is outlined here.

Graham's Variety

Graham has defined a new variety which maps to $\widetilde{\mathcal{N}}$. We'll denote this as $\widetilde{\mathcal{M}}$. The maps are denoted as

$$\gamma : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}} \quad \text{and} \quad \widetilde{\mu} = \mu \circ \gamma : \widetilde{\mathcal{M}} \to \mathcal{N}.$$

A key feature of Graham's variety is that it carries a nontrivial action of the center of the group G, while the center acts trivially on both $\widetilde{\mathcal{N}}$ and $\mathcal{N}$.

Generalized Springer Correspondence for SL_n (Graham–Precup–Russell)

Assume G is $SL_n(\mathbb{C})$. Let $\underline{\mathbb{C}}_{\widetilde{\mathcal{M}}}$ denote the constant sheaf on $\widetilde{\mathcal{M}}$. Then the Decomposition Theorem for Perverse Sheaves says

$$R\mu_* \underline{\mathbb{C}}_{\widetilde{\mathcal{M}}} = \bigoplus_{x,\varphi} \mathrm{IC}(\mathcal{O}_x, L_\varphi) \otimes V_{x,\varphi}$$

where $\mathrm{IC}(\mathcal{O}_x, L_\varphi)$ is a simple perverse sheaf, $V_{x,\varphi}$ is a representation of a relative Weyl group for G, and the (x, φ) are all possible pairs.